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Invasion percolation and invading Eden growth on multifractal lattices

Paul Meakin

Central Research and Development Department, E I du Pont de Nemours and Co Inc, Wilmington, DE 19898, USA

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Abstract. Eden growth and invasion percolation models have been explored on square lattices with a multifractal distribution of growth probabilities. These models generate structures with a mass fractal dimensionality of 2 and surfaces which can be described in terms of self-similar fractal geometry with a fractal dimensionality larger than 1 and smaller than 2. For the Eden growth models the fractal dimensions of the total perimeter, internal hull and external hull depend on the probabilities associated with the generator of the multifractal measure. For the invasion percolation model the three fractal dimensionalities are relatively insensitive to the structure of the generator and may be universal. The distribution of growth probabilities associated with the Eden models can also be described as a fractal measure and the spectrum of singularities $f(\alpha)$ associated with this measure has been estimated for some of these models. For the Eden growth models, the fractal dimensionalities describing the structure of total perimeter, internal hull and external hull are given by $D = d(\xi)$ where ξ is the variance of the logarithms of the probabilities used in the multiplicative generators for the multifractal growth probability measure. The function $d(\xi)$ seems to be the same for the internal and external hulls and quite different from $d(\xi)$ for the total surface.

1. Introduction

Since the introduction of the diffusion-limited aggregation (DLA) model by Witten and Sander (1981) considerable interest has developed in a wide variety of growth and aggregation models. In most of these models, such as the Eden (1961) model, the growth is assumed to take place on a uniform Euclidean or fractal (Mandelbrot 1982) lattice (see, e.g., Martin *et al* 1984). In a variety of other models such as invasion percolation (Wilkinson and Willemsen 1983, Wilkinson and Barsony 1984, Lenormand and Bories 1980, Chayes *et al* 1985) the growth probabilities depend on random numbers which have been selected from some distributions and assigned to the individual lattice sites. A family of models of this type have ben investigated by Martin *et al* (1984). Although many of these models are extremely simple to define, they often lead to surprising results and generate complex (often fractal) structures. A variety of examples can be found in recent reviews (Herrmann 1986, Jullien 1986, Witten and Cates 1986, Sander 1986a, b, Meakin 1987a, 1988b), books (Jullien and Botet 1987) and conference proceedings (Family and Landau 1984, Stanley and Ostrowsky 1986, Pynn and Skjeltorp 1986, Pietronero and Tosatti 1986b).

Much of the work on non-equilibrium growth models has been motivated by the need to obtain a better understanding of the physics of porous media for both scientific

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and practical purposes. In many porous systems (such as a bed of sand) the properties are quite uniform on all length scales larger than a characteristic correlation length (in the case of a bed of sand, this is approximately the diameter of the individual sand grains). In other systems, however (such as oil reservoirs (Hewett 1986)), the structure is non-uniform over a very wide range of length scales and strong correlations exist in the spatial distribution of physical and chemical properties. It has recently been suggested that, in some systems, it might be possible to describe this distribution of properties in terms of a fractal measure (Mandelbrot 1974, 1982, Halsey *et al* 1986a, b). In this picture the system is considered to be capable of being decomposed into subsets, each of which has associated with it a different scaling index α. The subset of sites associated with the scaling index α (which is a continuously variable quantity) has a fractal dimensionality $f(\alpha)$. This has motivated the development of growth models (Meakin 1987c, d) and the study of random walks (Meakin 1987e, Weissman and Havlin 1987) on lattices which have a fractal probability measure which controls the growth or random walk. The function $f(\alpha)$ can be obtained from the asymptotic (large range of length scales) distribution of probabilities associated with the fractal measure (Halsey *et al* 1986a, b, Meakin *et al* 1986a, b). In this event those sites with probability measures lying in the range μ to $\mu+\delta\mu$ (or $\ln(\mu)$ to $\ln(\mu)+\delta\ln(\mu)$) will lie on a fractal subset whose fractal dimensionality is given by $f(\alpha)$ (see Meakin (1988b) for a simple example). To my knowledge no quantitative experimental data exist which support this picture for the structure of porous systems (or other random structures). However, it provides one of the simplest possible models for the spatial correlation of properties in random systems and for this reason alone it merits further investigation. Here the results of an investigation of Eden growth and invasion percolation on square lattices with fractal measures generated using simple multiplicative hierarchical generators will be presented. The results of a preliminary investigation for one of these multifractal lattices have been presented recently (Meakin 1987c).

The Eden (1961) model is perhaps the simplest of all the non-equilibrium growth models. In this model unoccupied perimeter sites (empty sites with one or more occupied nearest neighbours) are filled randomly with probabilities which are proportional to their number of occupied nearest neighbours). Most simulations have been carried out using an even simpler version of the model in which all of the perimeter sites have equal growth probabilities. More recently, a third version of the Eden model (Jullien and Botet 1985a, b) has been extensively investigated. In this model occupied surface sites are randomly selected and one of the unoccupied nearest neighbours (also randomly selected) is then filled. This model seems to reduce the surprisingly large corrections to scaling associated with the surface structure of Eden clusters which has been a subject of considerable recent interest (Plischke and Racz 1984, Family and Vicsek 1985, Jullien and Botet 1985a, b, Freche *et al* 1985, Kardar *et al* 1986, Hirsch and Wolf 1986, Zabolitzky and Stauffer 1986, Meakin *et al* 1986a, Stauffer and Zabolitzky 1986). Despite the simplicity of these Eden models, which allows very large-scale simulations ($\simeq 10^{10}$ sites) to be carried out, uncertainty remains (particularly for dimensionalities greater than 2) concerning the quantitative aspects of the asymptotic surface structure (it has been known for over a decade (Richardson 1973) that the internal structure is compact). However, it has recently been shown that noise reduction (similar to that used earlier for DLA models (Szep *et al* 1985, Kertesz and Vicsek 1986, Tang 1985)) considerably reduces the corrections to scaling for the two-dimensional Eden model (Wolf and Kertesz 1987). Irrespective of the quantitative aspects of the surface structure of Eden cluster there now seems to be a concensus that the surface can be described as a self-affine fractal (Mandelbrot 1982, 1985, 1986, Voss 1986). It will be shown below that the surface structure of Eden clusters grown on multifractal lattices can be described in terms of self-similar fractal geometry.

The invasion percolation and Eden models are closely related (Martin *et al* 1984). In invasion percolation, that site on the unoccupied perimeter with the largest probability always grows. In a simple modification of the Eden model (used in our work) the growth probability associated with a particular perimeter site is equal to its probability measure, μ_i, or to some function of that measure (which defines a new measure). If the distribution of probabilities is very narrow, then this model is equivalent to the ordinary Eden model but, if the distribution becomes very broad, then this model becomes equivalent to invasion percolation. If the growth probability P_i at the ith site is given by $P_i = A\exp[-(\mu_i/\Delta)]$ then Δ can be thought of as a temperature (Martin *et al* 1984). In the low-temperature limit we have invasion percolation and in the high-temperature limit Eden growth. Except for the presence of spatial correlations, the models discussed in this paper are very similar to those introduced by Martin *et al* (1984).

2. Computer models

All of the simulations were carried out on two-dimensional square lattices of size $2^{10}\times 2^{10}$ lattice units. A fractal measure is generated on the lattice by first dividing the lattice into four quadrants (each of size $2^9\times 2^9$ lattice units) and randomly assigning four numbers (P_1, P_2, P_3 and P_4) with all of the lattice sites in each of the quadrants. Each of the quadrants is then divided into smaller quadrants of size 2^8 and the number associated with the lattice sites in each of the smaller quadrants is multiplied by P_1, P_2, P_3 or P_4 taken in random order for each subquadrant. The procedure outlined above is repeated 10 times in a hierarchical fashion until a number of the form $P_1^iP_2^jP_3^kP_4^l$ $(i+j+k+l=10)$ is associated with each of the lattice sites. This procedure generates a fractal measure μ which associates a number $\mu(x)$ with each of the positions on the lattice. A site is then randomly selected and the lattice with its measure is then translated, using periodic boundary conditions, so that the selected site is at the centre of the lattice. In some of the simulations the initial growth site or seed is selected with a probability proportional to the value of the measure $\mu(i,j)$ at that site.

In the Eden model the unoccupied perimeter sites (empty sites with one or more occupied nearest neighbours) are selected at random and the selected site is occupied if a random number X uniformly distributed in the range $0<X<1$ is smaller than $\mu^{(i,j)}/\mu_{\max}$. Here $\mu(i,j)$ is the value of the measure associated with the randomly selected perimeter site at position (i,j) and $\mu_{\max}$ is the maximum value of μ for any of the perimeter sites. The growth process is continued until the cluster either reaches a size of 100 000 sites or reaches the edge of the lattice. The different forms for the fractal measure on the substrate lattices were investigated. For model I, $P_1=1$, $P_2=R_1$, $P_3=R_1^2$ and $P_4=R_1^3$; for model II, $P_1=1$, $P_2=R_2$, $P_3=R_2$ and $P_4=R_2^2$ and for model III, $P_1=1$, $P_2=1$, $P_3=R_3$ and $P_4=R_3$. In each case the measure is determined by a single parameter (R_1, R_2 or R_3 for models I, II and III, respectively).

The invasion percolation model corresponds to the Eden model in the limit where $R(R_1, R_2$ or $R_3)\to 0$. In this case the simulation can be made considerably more efficient by selecting the next growth site randomly from a list of those sites which have the highest growth probability. Since the measure has the form $P_1^iP_2^jP_3^kP_4^l$ at each lattice site the histogram for the measure is discrete and there are only a few possible values for the growth probabilities (31 for model I, 21 for model II and 11 for model III) for the system size of 10 generations used here. In general, a number of sites (≥ 1) will have equal highest growth probabilities of 1.0, corresponding to the lowest value of m where the measure associated with each of the sites has the form R^m (for $R < 1$). The growth probabilities associated with all of the other perimeter sites will be zero ($R \to 0$).

For the Eden growth models the number of simulations which can be carried out using the algorithms outlined above in a reasonable amount of computer time (about 12 h of CPU time on an IBM 3090 computer) depends on the probability ratio R. For the values of R close to 1.0, 500–1000 clusters were grown. For the smallest values of R investigated in this work 30–60 clusters were grown. For the invasion percolation models 100–200 clusters were grown for each of the six models (models I, II and III with random and non-random origins) investigated.

3. Results

3.1. Eden models

In earlier work carried out using model I ($P_1 = 1$, $P_2 = R$, $P_3 = R^2$ and $P_4 = R^3$) results were obtained which indicated that the total surface (unoccupied perimeter sites), inner hull and outer hull could be described as self-similar fractals and that the effective fractal dimensionalities varied continuously with $R(R_1)$.

Figure 1(a) shows a 85 870 site cluster grown using model II with the parameter R_2 set to a value of 0.1. This cluster was obtained with a version of the model in which the growth site is selected randomly with a probability proportional to the value of the measure at that site. Using model II 599 clusters were grown using $R_2 = 0.8$, 242 clusters were grown with $R_2 = 0.4$, 116 clusters were grown with $R_2 = 0.2$, 45 clusters were grown with $R_2 = 0.1$ and 43 clusters were grown with $R_2 = 0.05$. The initial growth sites were selected randomly (irrespective of the value of the measure associated with each of the lattice sites). The fractal dimensionalities of the clusters were estimated from the dependence of the radius of gyration (R_g) on the cluster mass or number of occupied sites (M). Figure 2 shows the dependence of $\ln(R_g/M^{1/2})$ on $\ln(M)$ obtained from the simulations. The fact that these curves are essentially horizontal indicates that $R_g \sim M^{1/2}$ and that the fractal dimensionality is 2.0. If the slope of about 0.023 for $R_2 = 0.05$ is taken seriously, this would correspond to a fractal dimensionality of about 1.90. However, the results shown in figure 2 are also consistent with the idea that $D = 2.0$ for all values of R_2. Similar results were obtained earlier for model I (but were not published) and from model III.

Figure 3 shows the two-point density-density correlation function $C(r)$ for the total surface, internal hull and external hull obtained from the simulations used to generate the results shown in figure 2. The total surface consists of all of those unoccupied sites with one or more occupied nearest neighbours. The internal hull consists of all of the occupied sites in the cluster which can be connected to a site at infinity by means of a path which consists of steps to nearest-neighbour or next-nearest-neighbour sites without passing through an occupied site. The external hull consists of all of the unoccupied perimeter sites which can be reached from outside of the region occupied by the cluster by paths which connect only unoccupied nearest

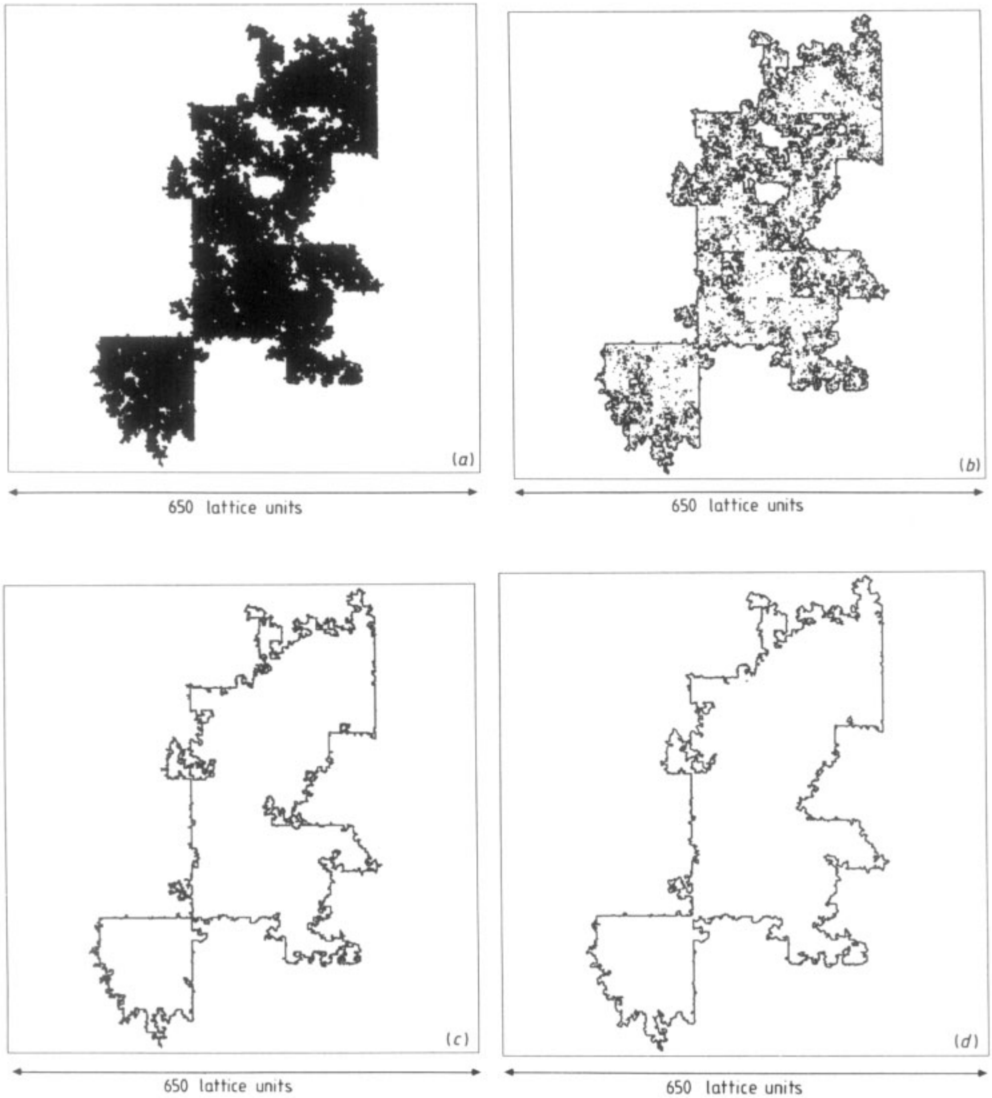

Figure 1. A cluster of 85 870 sites grown using model II. In this model the generator is given by $P_1 = 1$, $P_2 = 0.1$, $P_3 = 0.1$ and $P_4 = 0.01$ ($R_2 = 0.1$). (a) shows all the 85 870 occupied sites, (b) shows the 17 218 sites in the unoccupied perimeter, (c) shows the 8874 external hull sites and (d) shows the 5658 internal hull sites.

neighbours. For the invasion percolation model which seems to be quite closely related to the models investigated in connection with this work, the internal hull and external hull have different fractal dimensionalities (Meakin and Family 1986). For the case of site percolation on a lattice the outer hull has a fractal dimensionality of $\frac{7}{4}$ (Voss 1984, Ziff 1986, Sapoval *et al* 1985, Saleur and Duplantier 1987, Coniglio *et al* 1987) while the inner hull has a fractal dimensionality of $\frac{4}{3}$ (Grossman and Aharony 1986, Saleur and Duplantier 1987, Coniglio *et al* 1987). Since the two fractal dimensionalities are quite different for invasion percolation and site percolation it seemed worthwhile to measure both of them for the models studied here.

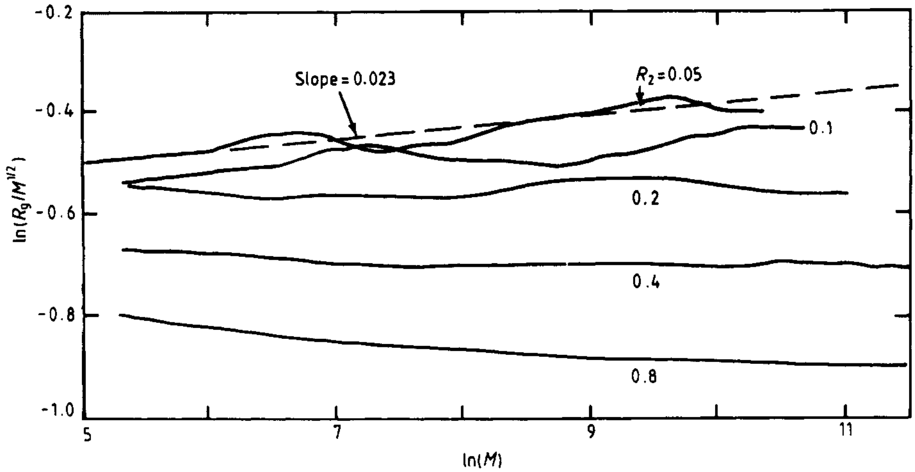

Figure 2. Dependence of $\ln(R_g/M^{1/2})$ on $\ln(M)$ for Eden growth model II with five different values for the probability ratio, R_2, which defines the generator for the multifractal measure. Here R_g is the radius of gyration and M is the cluster size (number of occupied sites).

Figure 3 shows that the dependence of $\ln[C(r)]$ on $\ln(r)$, where r is the distance, is essentially linear over a significant range of length scales. This indicates that the surfaces of these Eden growth clusters can be described in terms of self-similar fractal geometry. If the correlation function can be described by the power law form

$$C(r) \sim r^{\alpha} \tag{1}$$

then the corresponding fractal dimensionality D_α is given by $D_\alpha = d - \alpha$. Results obtained by least-squares fitting of straight lines to the coordinates ($\ln(r)$, $\ln[C(r)]$) are given in table 1.

A similar series of simulations was carried out using the related model in which the initial growth site was selected with a probability proportional to its (fractal) measure μ. Results from these simulations are also shown in table 1. The results given in table 1 indicate that both versions of model II give very similar results.

For model III ($P_1 = 1$, $P_2 = 1$, $P_3 = R_3$ and $P_4 = R_3$) a series of simulations was carried out with randomly selected growth sites. Six different values for R_3 (0.8, 0.4, 0.2, 0.1, 0.05 and 0.025) were used. The number of clusters generated was 641, 291, 190, 120, 67 and 57, respectively. The fractal dimensionalities (D_α) obtained from these simulations are shown in table 2 for all six values of R_3. It is apparent from the results shown in tables 1 and 2 and those obtained previously from model I that all three models give very similar results. Figure 4 shows the dependence of D_α on the parameter R_1 obtained using model I. It seems evident that, as $R \to 0$, the fractal dimensionalities of the total surface and the internal and external hulls approach limiting values smaller than 2.0. It also seems that both hulls may approach the same limiting fractal dimensionality and that these limiting values for D_α may be the same for all three models. The invasion percolation model was developed to explore these questions in more detail. Preliminary results have already been given for model I (from 129 clusters with randomly selected initial growth sites) which indicate that the fractal dimensionality (D_α) for the total surface approaches a limiting ($R \to 0$) value of 1.77 ± 0.01 and the

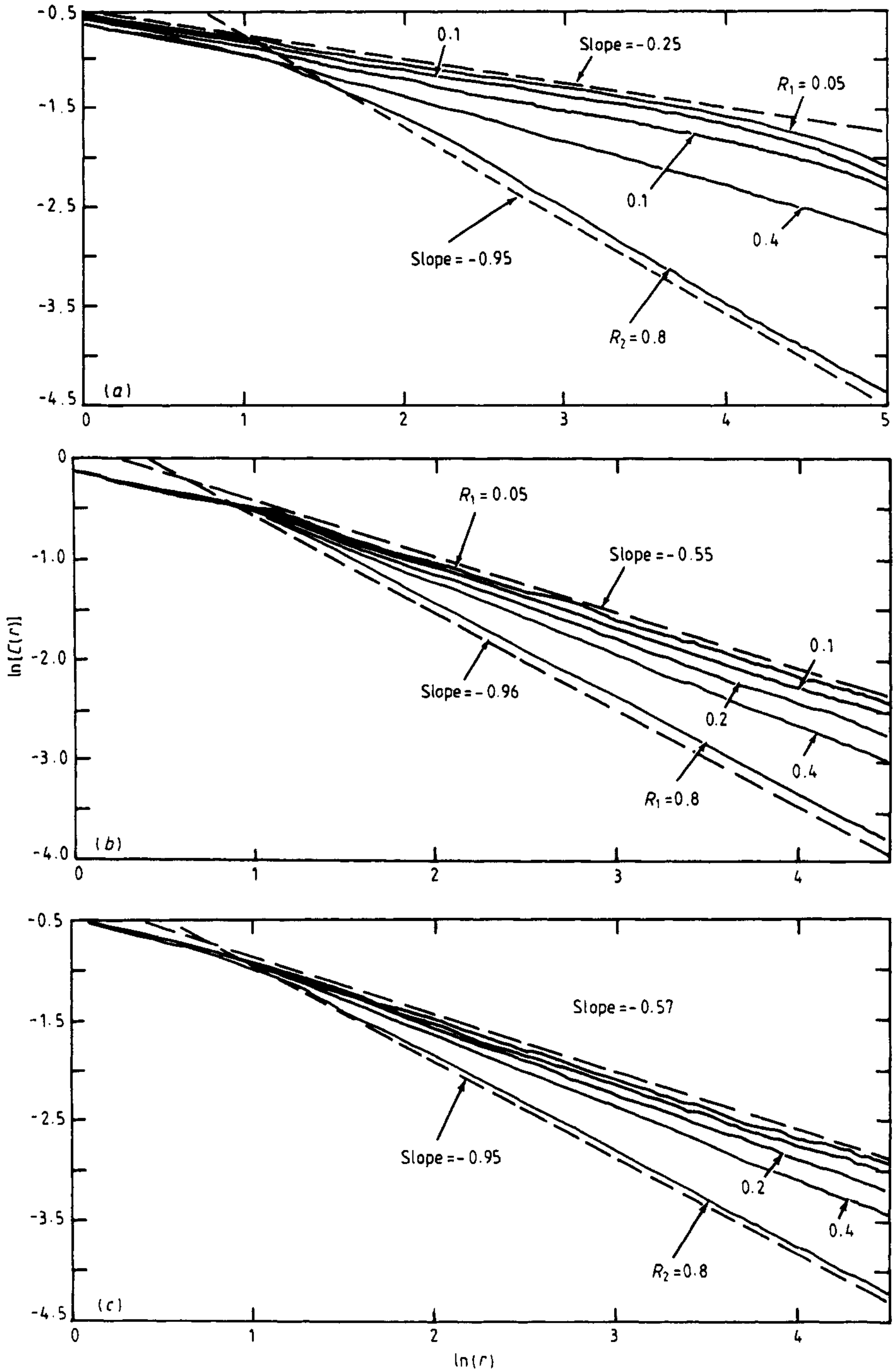

Figure 3. Density-density correlation functions for clusters generated using Eden model II. The correlation functions are shown for five values of the probability ratio R_2 (0.05, 0.1, 0.2, 0.4 and 0.8). (*a*) shows the correlation functions for the total surface (unoccupied perimeter). (*b*) and (*c*) show the correlation functions for the internal and external hulls, respectively.

Table 1. Dependence of the fractal dimensionality of the total surface, internal hull and external hull of Eden growth clusters generated using model II. These results were obtained from the two-point density-density correlation function $C(r)$ by least-squares fitting straight lines to the dependence of $\ln[C(r)]$ on $\ln(r)$ over the range of length scales (in lattice units) indicated. The first set of results (a) were obtained using randomly selected growth sites and the second set (b) were obtained from simulations in which the initial growth site was selected with a probability proportional to the measure associated with that site.

R_2	Total surface, $5 \leq r \leq 50$	Internal hull, $5 \leq r \leq 100$	External hull $5 \leq r \leq 100$
(a) 0.8	1.07	1.06	1.06
0.4	1.55	1.29	1.29
0.2	1.70	1.37	1.36
0.1	1.73	1.42	1.42
0.05	1.74	1.45	1.44
(b) 0.8	1.05	1.05	1.06
0.4	1.55	1.29	1.28
0.2	1.70	1.38	1.38
0.1	1.73	1.43	1.42
0.05	1.74	1.43	1.42

Table 2. Fractal dimensionalities obtained for the total surface, internal hull and external hull for clusters grown on a multifractal lattice using model III. In these simulations the initial growth site was selected randomly, irrespective of the value of the measure associated with it. The fractal dimensionalities (D_α) were obtained from the two-point density-density correlation functions in the same way as those reported in table 1.

R_3	Total surface $5 \leq r \leq 50$	Internal hull $5 \leq r \leq 100$	External hull $5 \leq r \leq 100$
0.8	1.03	1.03	1.04
0.4	1.40	1.21	1.21
0.2	1.62	1.33	1.32
0.1	1.71	1.38	1.37
0.05	1.72	1.42	1.41
0.025	1.74	1.44	1.44

dimensionalities of the internal and external hulls approach values of 1.47 ± 0.01 and 1.43 ± 0.01, respectively.

3.2. *Invasion percolation models*

An invasion percolation cluster grown on a model II ($P_1 = 1$, $P_2 = R_2$, $P_3 = R_2$, $P_4 = R_2^2$) multifractal lattice is shown in figure 5(a). The cluster contains 72 583 sites. Figure 5(b) shows the 22 511 unoccupied perimeter sites and figure 5(c) shows the location of the 17 639 sites in the internal hull. The 10 980 site external hull is shown in figure 5(d). This figure was generated using a model in which the growth started at the site with the largest value of the measure (in this model there is just one of these sites for which the measure has a value of 1). Figure 6 shows the density-density correlation function for the total perimeter (figure 6(a)), internal hull (figure 6(b)) and external hull (figure 6(d)) of the clusters generated using this model. Similar results for model

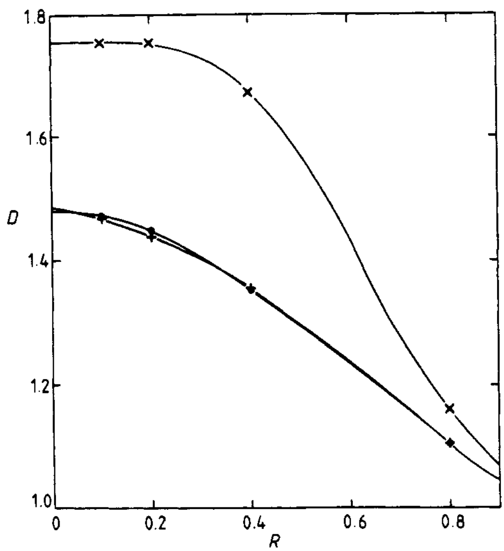

Figure 4. Dependence of the effective fractal dimensionalities describing the total surface (×), internal hull (●) and external hull (+) for clusters generated using Eden model I. Here the dependence of the fractal dimensionalities on the generator for the fractal measure on the substrate is shown. This generator is specified by $P_1 = 1$, $P_2 = R_1$, $P_3 = R_1^2$, $P_4 = R_1^3$. In this figure R indicates R_1.

I and model III multifractal substrate lattices are also shown. A total of 74 clusters for model I, 133 clusters for model II and 102 clusters for model III were used to obtain these correlation functions.

Simulations were also carried out using the invasion percolation model with randomly selected initial growth sites. 183 clusters were grown on model I substrates, 183 clusters were grown on model II substrates and 193 clusters were grown on model III substrates. The correlation functions for the total surface, interior percolation hull and exterior percolation hull are shown in figure 7. The values of the fractal dimensionality (D_α) obtained from these correlation functions are shown in table 3.

The fractal dimensionality of the clusters themselves (the 'mass' dimensionality) has been estimated from the dependence of $\ln(R_g/M^{1/2})$ on $\ln(M)$. The results obtained from all six models are shown in figure 8. These results indicate that the fractal dimensionality D_β obtained, assuming that $R_g \sim M^\beta$ and that $D_\beta = 1/\beta$, is approximately 2.0 for all models. The dependence of R_g on M does not exclude a fractal dimensionality slightly lower than 2.0 but values as low as 1.9 are inconsistent with the data displayed in figure 8. It seems most probable that the fractal dimensionality for these clusters is 2.0.

4. The growth probability measure

It has recently been shown (Halsey *et al* 1986a, b, Meakin *et al* 1985, 1986a, b, Meakin 1986, 1987b, Amitrano *et al* 1986, Pietronero and Tosatti 1986a) that the distribution of growth probabilities associated with a variety of processes leading to the formation of fractal structures can be described in terms of a fractal measure and its associated

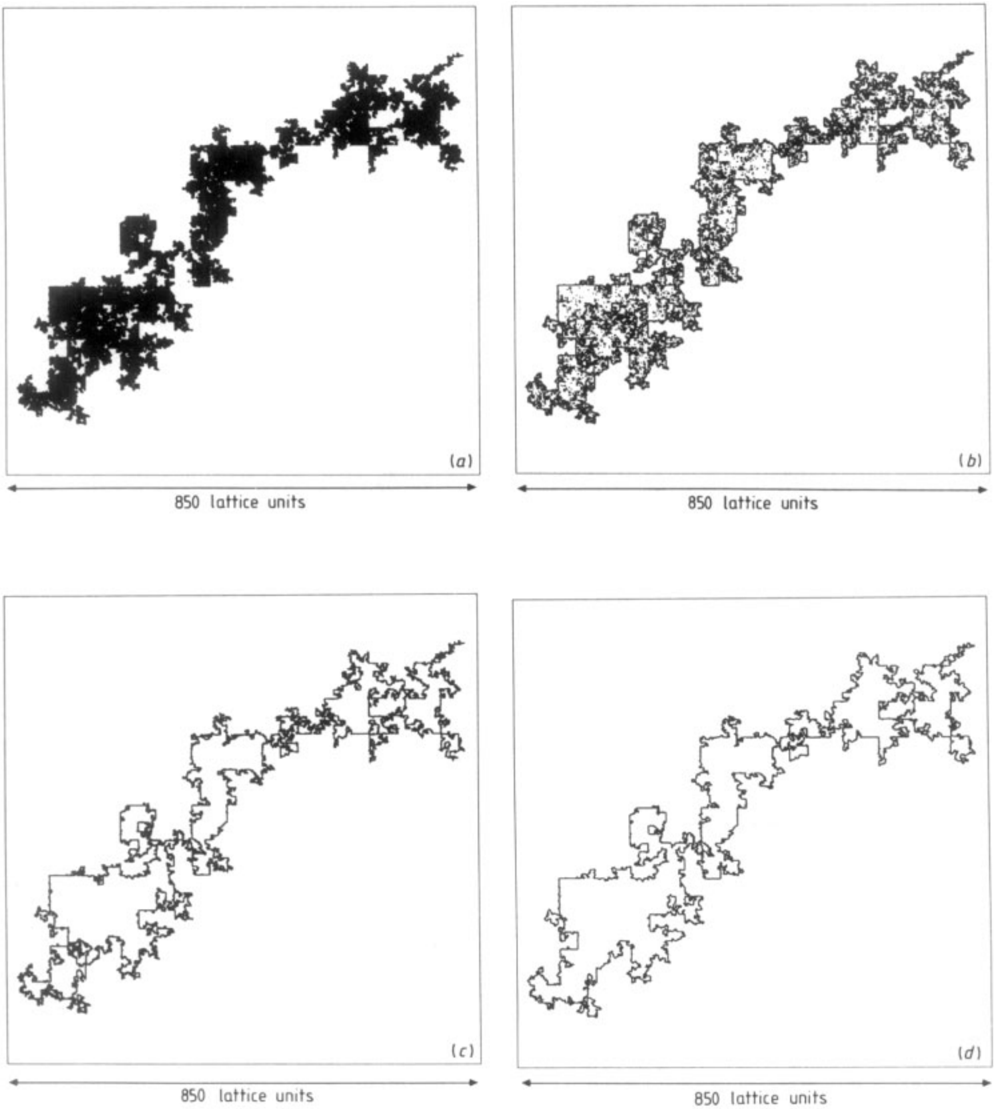

Figure 5. A 72 583 site cluster grown using invasion percolation model II. The growth of this cluster was started at the site with the largest value of the fractal measure. (*a*) shows all of the occupied sites, (*b*) shows the 22 511 unoccupied perimeter sites, (*c*) shows the 17 639 internal hull sites and (*d*) shows the 10 980 external hull sites.

infinite family of fractal dimensionalities or scaling exponents (Mandelbrot 1974, Halsey *et al* 1986a, b, Hentschel and Procaccia 1983, Grassberger and Procaccia 1983, Benzi *et al* 1984). In view of this work and the way in which the models investigated in this paper were constructed, it seems reasonable to expect that they too should have a multifractal distribution of growth probabilities. To test this idea a record was kept of all of the growth probabilities associated with all of the unoccupied perimeter sites at eight stages during the growth of the structures described above. These eight stages correspond to cluster masses of $M = 1000, 2000, 4000, 8000, 16\,000, 32\,000, 64\,000$ and $100\,000$.

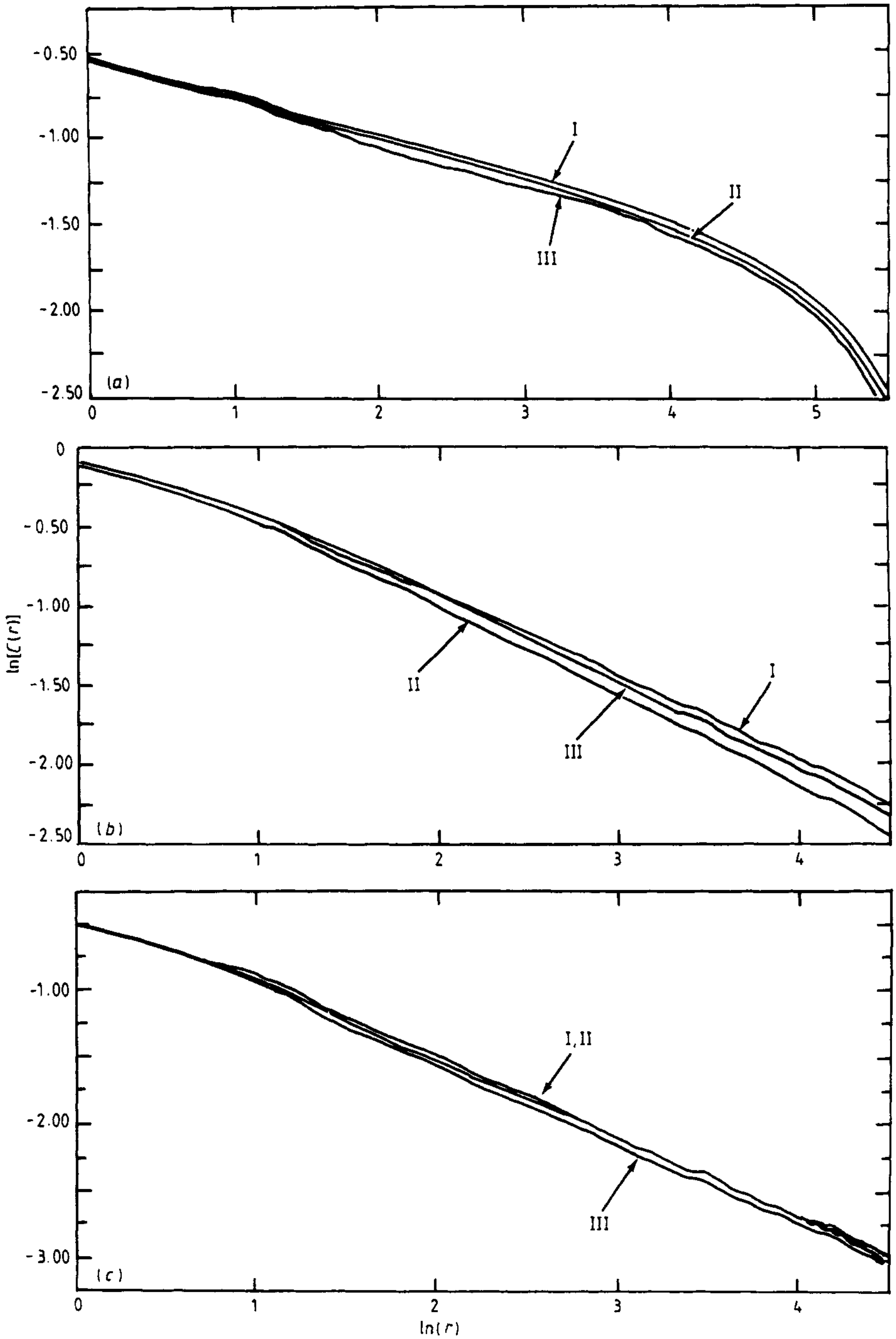

Figure 6. Two-point density-density correlation functions for the surfaces of invasion percolation models I, II and III. (*a*) shows the correlation functions for the unoccupied perimeters, (*b*) shows the correlation functions for the internal hulls and (*c*) shows the correlation functions for the external hulls. These correlation functions are the average correlation functions for a large number of simulations, each of which was started at the site with the highest value of the fractal measure.

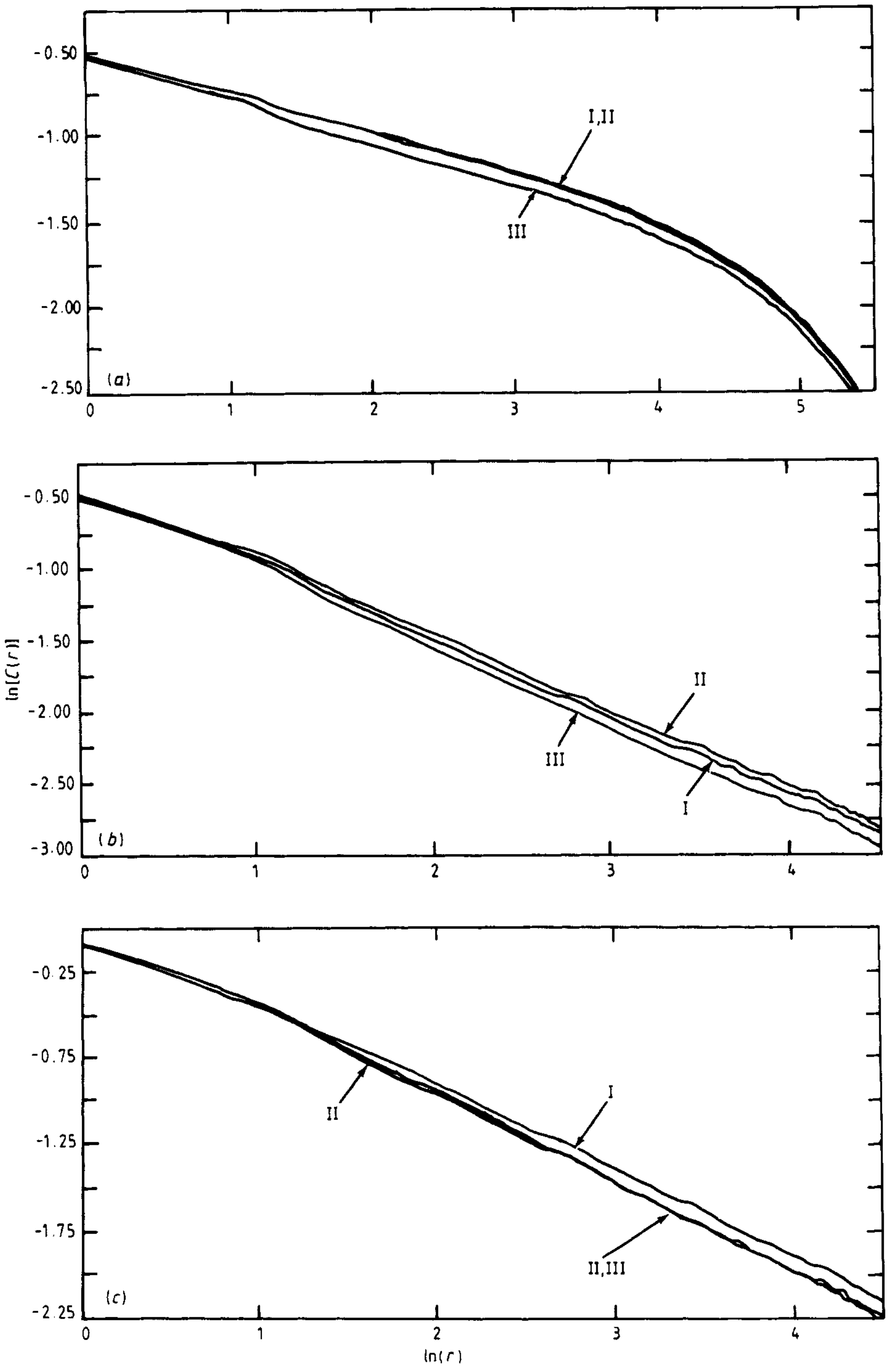

Figure 7. Correlation functions for the invasion percolation model surfaces. Results are shown for all three models. The results shown here were obtained from simulations very similar to those used to obtain figure 6 except that the growth was started at random positions, irrespective of the multifractal measures on the lattice.

Table 3. Values of the fractal dimensionality (D_α) obtained from the two-point density-density correlation functions shown in figures 6 and 7. These results were obtained from the correlation function over the range $4 \le r \le 40$ lattice units for the total surface and $5 \le r \le 100$ lattice units for the internal and external hulls. The results shown in (a) were obtained with randomly selected growth sites and those in (b) from simulations in which the growth started from the site with the largest value for the fractal measure (models I and II) or one of the sites with the largest value of the measure (model III).

	Model	Total surface	Internal hull	External hull
(a)	I	1.757	1.501	1.467
	I	1.761	1.507	1.474
	I	1.758	1.498	1.462
	I	1.755	1.493	1.465
	II	1.76	1.48	1.47
	III	1.75	1.49	1.46
(b)	I	1.78	1.48	1.43
	II	1.77	1.42	1.40
	III	1.76	1.46	1.42

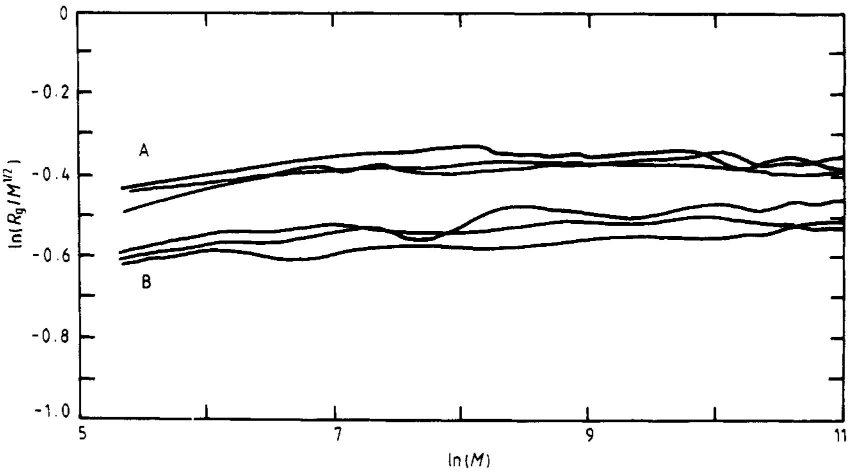

Figure 8. Dependence of $\ln(R_g/M^{1/2})$ on $\ln(M)$ for the invasion percolation models. The curves in group A were obtained from models I, II and III with randomly selected growth sites. The curves in group B were obtained from models I, II and III with growth originating at the site with the highest value of the fractal measure.

For each cluster generated with a particular set of growth parameters, the quantity $Z(q)$ defined by

$$Z(q) = \sum_{i=1}^{N_s} P_i^q \tag{2}$$

was determined over a range of positive and negative values of q (typically $Z(q)$ would be measured in the range of about $q = -2$ to $q = 10$ at intervals of 0.1 in q and this is the range and interval used unless otherwise indicated). In equation (2) the probabilities P_i are normalised growth probabilities ($\Sigma\, P_i = 1$). If the growth probabilities constitute a fractal measure, then we expect that the quantity $Z(q)$ will scale with an overall size of the system (L) according to

$$Z(q) \sim L^{-\tau(q)} \tag{3}$$

and in the asymptotic limit ($M \to \infty$ or $L \to \infty$) the exponent $\tau(q)$ is given by

$$\tau(q) = \log(Z(q))/\log(L). \tag{4}$$

For finite-size systems the value obtained for $\tau(q)$ will depend on which length (L) is used (radius of gyration, maximum radius, $M^{1/D}$, etc).

If $\tau(q)$ is known as a function of q, then the spectrum of singularities $f(\alpha)$ (Halsey *et al* 1986a,b) can be determined from the relationships

$$\alpha(q) = \mathrm{d}(\tau(q))/\mathrm{d}q \tag{5}$$

and

$$f(\alpha(q)) = q\alpha(q) - \tau(q). \tag{6}$$

To determine $f(\alpha)$ in this fashion for the models used in this work, $Z(q)$ was calculated individuallly for all of the clusters generated with a particular set of parameters and then averaged. Figure 9 shows some results obtained for Eden growth generated with model I using the parameters $P_1 = 1.0$, $P_2 = 0.8$, $P_3 = 0.64$ and $P_4 = 0.512$ ($R_1 = 0.8$). Figure 9(a) shows the $f(\alpha)$ curves obtained for the eight different growth stages ($M = 1000, 2000, \ldots, 100\,000$). In this case the quantity $M^{1/2}$ was used for the length L. The procedure outlined above always gives a smooth convex curve and does not convey any idea about the quality of the data or indicate if the growth probability measure is really a multifractal or not. Only by repeating the calculation of $f(\alpha)$ for different values of L can we determine if the measure is indeed a fractal measure and if the correct length L has been chosen. In figure 9(a) a different $f(\alpha)$ curve is obtained for each cluster size. Figures 9(b) and (c) show the $f(\alpha)$ curves obtained using length (L) of $2M^{1/2}$ and $4M^{1/2}$, respectively. Figure 9(c) shows that, if $4M^{1/2}$ is used for L, very similar $f(\alpha)$ curves are obtained for each cluster mass (except for $M = 1000$ sites). This scaling collapse of effective $f(\alpha)$ curves for different cluster masses indicates that the growth probability measure is a fractal measure and figure 9(c) provides an estimate of the shape of the asymptotic $f(\alpha)$ curve.

According to the picture of Halsey *et al* (1986a) for the structure of multifractal measures $f(\alpha)$ should be a smooth convex curve with a maximum value equal to the fractal dimension for the support of the measure. In figure 9(c) the maximum value of $f(\alpha)$ is about 1.15. For this model (with $R_1 = 0.8$) the fractal dimensionality of the total surface was measured (Meakin 1987e) and found to have a value of 1.12, in reasonably good agreement with the maximum value of $f(\alpha)$. Figure 10(a) and (b) show effective $f(\alpha)$ curves obtained from model I with $R_1 = 0.4$ and 0.2, respectively. The maximum values of $f(\alpha)$ are approximately 1.45 for $R_1 = 0.4$ and 1.75 for $R_1 = 0.2$. The fractal dimensionality of the total perimeters are 1.35 and 1.75 for $R_1 = 0.4$ and $R_1 = 0.2$, respectively. The reasonably good scaling collapse of $f(\alpha)$ for different cluster sizes covering two orders of magnitude in M and one order of magnitude in L combined with the quite good agreement of the maximum value of $f(\alpha)$ with the fractal dimensionality of the growing surface provide strong evidence for a multifractal growth process. Despite the quite good data collapse shown in figure 9(c) the curves in this figure may differ substantially from the asymptotic $f(\alpha)$. The x and y coordinates of each point on the curves shown in figure 9(c) correspond to the effective values of two exponents

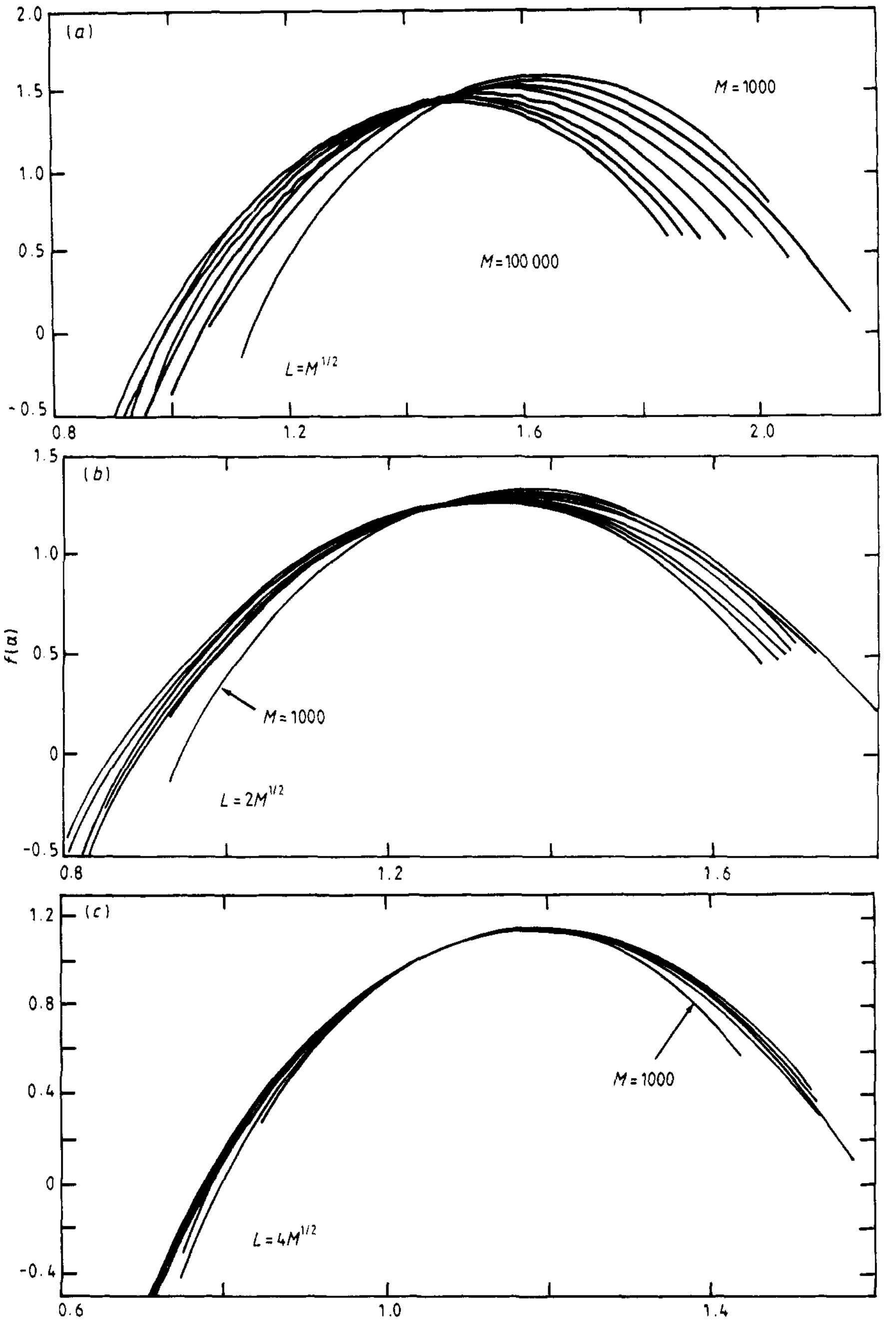

Figure 9. Effective $f(\alpha)$ curves for Eden model I and the generators $P_1 = 1$, $P_2 = 0.8$, $P_3 = 0.64$ and $P_4 = 0.512$ ($R_1 = 0.8$). The spectrum of singularities was obtained for the growth probability measure on the perimeter sites using equations (1)-(5). (a)-(c) show the results obtained using three different measures of the overall cluster sizes (L). The best data collapse of the $f(\alpha)$ curves for different masses onto a single curve is shown in figure 8(c) ($L = 4M^{1/2}$). (a)-(c) were obtained from $Z(q)$ (equation (1)) with q in the range $-5 \leq q \leq 10$.

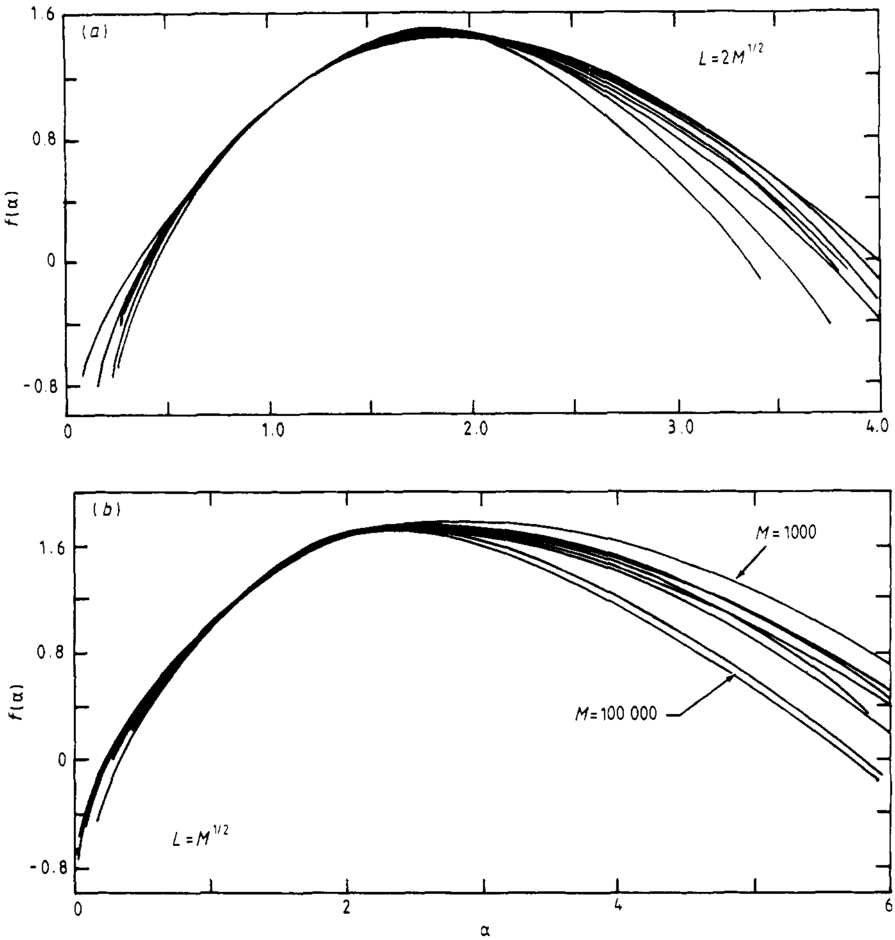

Figure 10. Effective $f(\alpha)$ curves for the growth probability measure associated with Eden model I for $R_1=0.4$ (a) and $R_1=0.2$ (b). These figures were obtained using $Z(q)$ in the range $-2 \leq q \leq 10$.

or scaling indices (α and $f(\alpha)$). The selection of the length L which gives the best data collapse does not remove all of the finite-size corrections. Other procedures for estimating the $f(\alpha)$ curve from the growth probability distribution could have been used (Meakin 1988a). However, at the present time very little is known about the nature of the corrections to scaling for $f(\alpha)$ and no general procedures for reducing their effects have been developed.

Figures 11(a) and (b) show results obtained from model II ($R_2=0.2$) and model III ($R_3=0.1$). Here the maximum values of $f(\alpha)$ are approximately 1.45-1.50 in both cases. The agreement with the fractal dimensionality of the total perimeter is not as good (a value of about 1.7 was obtained in both cases—table 1).

In the procedure used to estimate the $f(\alpha)$ curves shown in figures 9-11 it is implicitly assumed that the partition function $Z(q)$ can be written as

$$Z(q) \sim AL^{-\tau(q)} \tag{7}$$

where A is a constant prefactor. The value of A is that which gives the best data collapse for the effective $f(\alpha)$ curves obtained with different values of L. If a constant

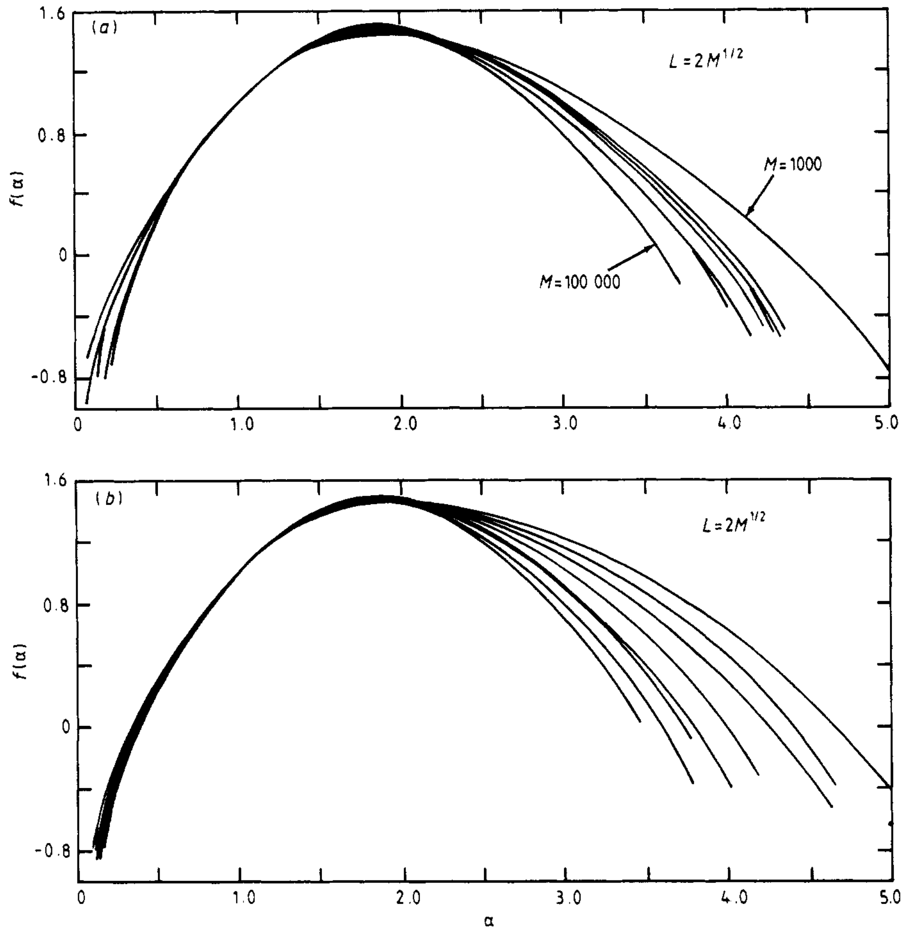

Figure 11. Effective $f(\alpha)$ curve for Eden models II and III. (a) shows the results for model II with $R_2 = 0.2$ and (b) show results for model III with $R_3 = 0.1$.

value is used for A, then the function $\tau(q)$ is convex and a complete $f(\alpha)$ curve can be obtained from $\tau(q)$ over a sufficiently broad range of q values. It would be more realistic to replace equation (7) by

$$Z(q) = A(q)L^{-\tau(q)} \tag{8}$$

and determine $\tau(q)$ from a least-squares fit of a straight line to the dependence of $\log(Z(q))$ on $\log(L)$ for many values of q. The procedure has been carried out using the data from which figures 9-11 were obtained. The effective $\tau(q)$ curves obtained in this way are not everywhere convex and a sensible $f(\alpha)$ curve can only be obtained for small values of α corresponding to those regions in which the growth probabilities are large. In this region the $f(\alpha)$ curves obtained from this procedure are quite similar to those shown in figures 9(c), 10 and 11.

In principle, corrections to this simple scaling picture should also be considered and equation (8) must be replaced by

$$Z(q) \sim A(L, q)L^{-\tau(q)}. \tag{9}$$

However, more elaborate approaches to the determination of $f(\alpha)$ do not seem to be warranted in view of the results obtained using equation (8).

5. Discussion

One of the main objectives of this work was to determine if the fractal dimensionalities characterising the surfaces of the clusters generated by Eden growth on multifractal lattices in the limit $R \to 0$ depends on the details of the generator used to construct the multifractal. The results of this aspect of the work are summarised in table 3. It is, in principle, not possible to establish universality on the basis of computer simulations alone and the results shown in table 3 seem to be particularly ambiguous. The differences in the fractal dimensionalities describing the surface structure are quite small (particularly if clusters with random origins are compared with clusters with random origins and those with origins at sites with the highest probability measures are compared with clusters generated using different generators but with growth originating from the sites with the highest probability measure). However, these differences seem somewhat larger than might be expected for large numbers of large clusters. Because of the complex structure of the multifractal substrate, the statistical uncertainties may be larger than expected. To get some idea of these uncertainties the simulations for invasion percolation with model I and random origins were repeated four times. The results from these simulations indicate that the statistical uncertainties are at least as large as the differences observed between models I, II and III using the same sort of growth origin. The fractal dimensionalities obtained for growth from the site with the highest value of the measure seem to be consistently smaller than those obtained with random origins. This result is intuitively reasonable since growth from the most probable site would be expected to give more compact structures than growth from a randomly selected site. However, it is not clear if this difference should be sufficient to change the asymptotic fractal dimensionality.

One way of characterising the multifractal measures used in this work is through their moments or the asymptotic form of their histograms. Either of these quantities can be used to obtain the spectrum of singularities $f(\alpha)$ (Halsey *et al* 1986a) which characterise the fractal measure. For the fractal measures used in this work the quantities P_1, P_2, P_3 and P_4 defining the generator can be regarded as probabilities and the generator is also defined by the probability ratios R_1, R_2 or R_3. The probability measure at each of the sites has the form R^m. The same measure could also be obtained using an additive hierarchical generator with the elements 0, 1, 2, 3 for model I, 0, 1, 1, 2 for model II and 0, 0, 1, 1 for model III. This would then generate the value of the exponent m at each site which would give the value for the measure R^m at that site. From the central limit theorem the additive generators corresponding to models I, II and III should lead to normal distributions which are characterised by the mean value and variance of the generator. The mean values are $\frac{3}{2}$, 1 and $\frac{1}{2}$, respectively, for models I, II and III and the variance σ is $\sqrt{5}/2$, $1/\sqrt{2}$ and $\frac{1}{2}$, respectively. This does not mean that the $f(\alpha)$ curve for the substrate should be a log-normal function. For model II, for example, $f(\alpha)$ should be log-binomial. The log-binomial and log-normal functions differ substantially for large and small values of α. However, near to its peak $f(\alpha)$ should have a log-normal shape. It has been shown in earlier work that growth probability on the surface of a growing Eden cluster has, on average, a much narrower distribution of growth probabilities than that associated with the substrate.

Consequently, it is reasonable to suppose that it is the central part of the $f(\alpha)$ curve for the substrate which will control the Eden growth process so that the variance of the additive generator will control the structure of Eden clusters grown on a multifractal substrate. If these ideas are correct, then simulations with models I, II and III with R_1, R_2 and R_3 given by $R_2 = R_1\sqrt{2}/\sqrt{5}$ and $R_3 = R_1/\sqrt{5}$ should give clusters with the same surface fractal dimensionalities. For example, if $R_1 = 0.5$, then simulations carried out with model II and $R_2 = 0.334\,22$ and with model III and $R_3 = 0.212\,26$ should give very similar results. Figure 12 compares the two-point density-density correlation functions carried out using these three models. The results shown in this figure were obtained from 465 simulations for model I, 153 simulations for model II and 235 simulations for model III.

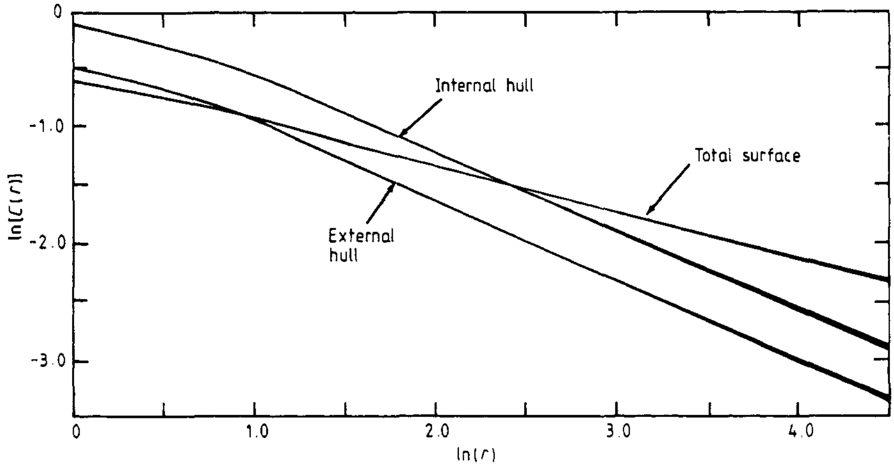

Figure 12. A comparison of the two-point density-density correlation functions for the total surface, internal hull and external hull obtained using model I with $R_1 = 0.5$, model II with $R_2 = 0.334\,22$ and model III with $R_3 = 0.212\,26$. These parameters correspond to $\sigma \ln(R) = -0.775$ where σ is the variance of the generator for the additive measures μ_a from which the multifractal (multiplicative) growth probability measure μ_p can be obtained from $\mu_p = R^{\mu_a}$. The results from these three sets of simulations cannot be distinguished on the scale used in this figure. However, the differences between them do result in a thickening of the curves towards the right-hand side.

Another way of testing this idea is to look at the dependence of the fractal dimensionalities of the total surface, internal hull and external hull on $\sigma \ln R$. Figure 13 shows such a plot using the data given in tables 1 and 2(a) and the results obtained earlier from model I (with randomly selected growth sites). The results shown in this figure indicate that $D = f(\sigma \ln(R))$ where the function f is the same for all three models. The function $f(x)$ is very similar for both the internal and external hulls. The fact that the results from all three models can be represented in this way provides strong evidence for the idea that $\sigma \ln(R)$ determines universality classes for all three models. The quantity $\sigma \ln(R)$ is the variance μ in the logarithms of the probabilities used in the multiplicative generators for the growth probability measure on the total lattice. This quantity has a value of 0 for ordinary Eden and infinity for the invasion percolation models.

The models investigated in this work are related to the invading Eden model of Martin *et al* (1984). In this model the growth probability P_i at the ith perimeter site

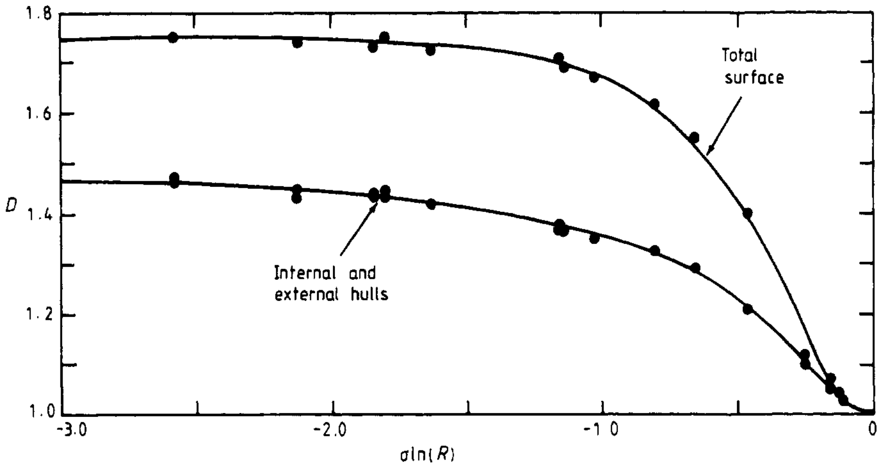

Figure 13. Dependence of the fractal dimensionalities of the total surface, inner hull and outer hull for Eden models I, II and III (with randomly selected growth sites) on $\sigma \ln(R)$. These results were obtained from four simulations using model I, five simulations using model II and six simulations using model III. A quite large number (about 100) of clusters were generated in each simulation. The actual data points are represented by large dots. In a few cases these dots cannot be resolved on the scale of this figure.

is given by

$$P_i = A \exp(-r_i/\Delta) \tag{10}$$

where r_i is a random variable and the parameter Δ can be regarded as a temperature. In the models used in this work the growth proabilities are given by

$$P_i = R^{M_i} \tag{11}$$

where M_i is a random variable obtained from the additive generator for the growth probability measure. Consequently, in these models $\ln(R)$ plays a role similar to $-1/\Delta$ in the invading Eden model. However, the results obtained from the models investigated here are very different from those obtained from the invading Eden model ($D=2$ for $\Delta>0$ and $D=D_p$ (the fractal dimensionality of a percolation cluster) for $\Delta=0$ with a crossover from a percolation cluster-like structure on short length scales to a uniform structure on long length scales for small non-zero values of Δ) because of the spatial correlations associated with the random variable M_i in equation (11).

One of the main conclusions of this work and of the preliminary investigation of these models (Meakin 1987c) is that the surfaces of the clusters are self-similar fractals. For the Eden model itself (Family and Vicsek 1985), and presumably for all variants of the Eden model with a finite ratio between the maximum and minimum growth probabilities, the surface can be described in terms of self-affine (Mandelbrot 1982, 1986, Voss 1986) fractal geometry. For models with a large but finite ratio between the maximum and minimum growth probabilities (Rikvold 1982, Meakin 1983) the surface structure (as well as the structure of the cluster itself) can appear to be self-similar even though in the asymptotic (large cluster size) limit the mass fractal dimensionality of the cluster is equal to that of the embedding space or lattice ($D=d$) and the surface has a self-affine fractal geometry with a global fractal dimensionality (Mandelbrot 1986) of $d-1$. For the models studied here the ratio between the maximum and minimum growth probability diverges algebraically as the cluster size increases. Consequently, these models are quite different from the previously studied Eden models and formation of a self-similar rather than a self-affine fractal surface does not appear to be unreasonable. It is also possible that, as in some other non-equilibrium growth models (Meakin and Vicsek 1985), the correlations may be different in the radial and tangential directions. The possibility that two or more scaling exponents (fractal dimensionalities) are needed to describe the surface structure has not yet been explored.

In some respects the models investigated here are more closely related to Eden growth on an incipient infinite percolation cluster (Family and Vicsek 1985). In this case the inner and outer hulls are self-similar fractals with different fractal dimensionalities (Grossman and Aharony 1986).

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